You can check if (*) and (**) are hold.

(*) ... $\mathcal{G}^{+} \vec{\underset{\sim}{x}} \mathcal{G} = \tilde{\underset{\sim}{x}}' = \tilde{\underset{\sim}{x}}$: obvious. (The position operator is the position operator.)

(**) ... $\mathcal{G}^{+} (\vec{\underset{\sim}{p}} - \frac{e}{c}\vec{A} - \frac{e}{c}\nabla\Lambda)\mathcal{G} = \tilde{\underset{\sim}{p}} - \frac{e}{c}\vec{A}$

· $\vec{A}(\underset{\sim}{\vec{x}})$, $\Lambda(\underset{\sim}{\vec{x}})$ commute with $\mathcal{G}(\underset{\sim}{\vec{x}})$.

· $$e^{-\frac{ie\Lambda}{\hbar c}} \vec{\underset{\sim}{p}}\, e^{\frac{ie\Lambda}{\hbar c}} = e^{-\frac{ie\Lambda}{\hbar c}} [\vec{\underset{\sim}{p}}, e^{\frac{ie\Lambda}{\hbar c}}] + \vec{\underset{\sim}{p}}$$

$$= e^{-\frac{ie\Lambda}{\hbar c}} (-i\hbar\nabla) e^{\frac{ie\Lambda}{\hbar c}} + \vec{\underset{\sim}{p}}$$

$$= \vec{\underset{\sim}{p}} + \frac{e}{c}\nabla\Lambda(\underset{\sim}{\vec{x}})$$

$$\Rightarrow \mathcal{G}^{+}(\vec{\underset{\sim}{p}} - \frac{e}{c}\vec{A} - \frac{e}{c}\nabla\Lambda)\mathcal{G} = \vec{\underset{\sim}{p}} + \cancel{\frac{e}{c}\nabla\Lambda} - \frac{e}{c}\vec{A} - \cancel{\frac{e}{c}\nabla\Lambda}$$

(try with H by yourself : $\mathcal{G}^{+}H\mathcal{G}$) #.

Indeed, $|\alpha'\rangle = \exp\left[\frac{ie}{\hbar c}\Lambda(\underset{\sim}{\vec{x}})\right]|\alpha\rangle$.

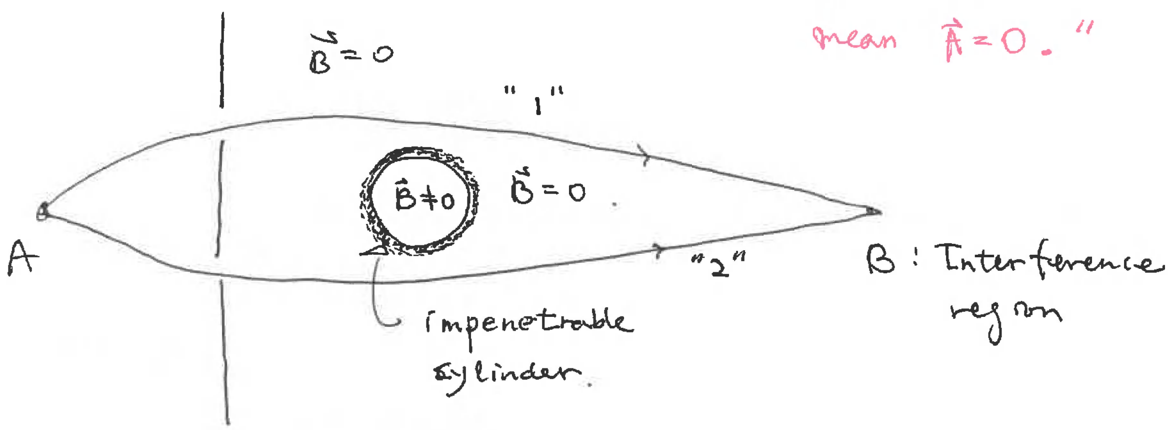
: The gauge transformation, $\vec{A} \to \vec{A} + \nabla\Lambda$,

introduces an extra phase factor ! in $\psi(x)$;

by changing $\vec{A}$ ($\vec{B}$ = same), one may expect some interferences due to the difference bet. the accumulated phases.

· Example 1: The Aharonov-Bohm effect.

" $\vec{B}=0$ does not necessarilly mean $\vec{A}=0$. "



now, consider the propagator of $A \to B$,

$$K(B,A) = \int_A^B \mathcal{D}[x(t)] \exp\left(\frac{i}{\hbar} S[x(t)]\right),$$

where the classical action $S[x(t)] = \int_{t_A}^{t_B} dt\, L(x, \dot{x}; t)$

– In the presence of $A(\vec{x})$, (let $\phi = 0$)

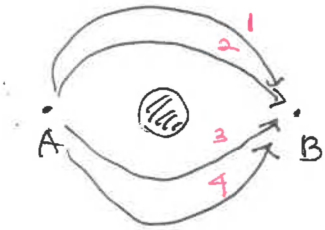
$$L = \frac{1}{2} m \dot{\vec{x}}^2 + \underbrace{\frac{e}{c} \dot{\vec{x}} \cdot A(\vec{x})}$$

: let's separate this term.

$$\Rightarrow K(B,A) = \int_A^B \mathcal{D}[x(t)] \overset{\exp(\frac{i}{\hbar}\int dt \frac{1}{2}m\dot{x}^2)}{\vee} \exp\left(\frac{ie}{\hbar c} \int_{t_A}^{t_B} dt \frac{d\vec{x}}{dt} \cdot \vec{A}(\vec{x})\right)$$

$$= \int_A^B \mathcal{D}[x(t)] \underbrace{\exp\left[\frac{i}{\hbar} S_0\right]}_{\text{for } \vec{A}=0} \cdot \underbrace{\exp\left(\frac{ie}{\hbar c} \int_{[x(t)]} d\vec{x} \cdot \vec{A}(\vec{x})\right)}.$$!!!

– Evaluation of $\int_\ell d\vec{x} \cdot \vec{A}(\vec{x})$ along a possible path ℓ:



Paths 1 and 2 : above $\vec{B}$ (or, detouring clockwise)

Paths 3 and 4 : below $\vec{B}$ (counterclockwise).

$$\Rightarrow \int_{\ell_1} d\vec{x} \cdot \vec{A}(\vec{x}) = \int_{\ell_2} [\cdots], \quad \int_{\ell_3} d\vec{x} \cdot \vec{A}(\vec{x}) = \int_{\ell_4} [\cdots].$$

Proof.

$$\int_{\ell_2} - \int_{\ell_1} = \oint_{\ell_2 - \ell_1} d\vec{x} \cdot \vec{A}(\vec{x}) :$$

$$= \int d\vec{\sigma} \cdot \vec{B} = 0. \quad \text{(Stokes' theorem)}$$

the same holds for $S_3 - S_4 = 0$.

But.

$$\underline{\int_3 - \int_2} \neq 0 :$$

$$= \underline{\int d\vec{\sigma} \cdot \vec{B} = \Phi_B} \quad \text{(magnetic flux)}$$

= const. for a given $\vec{B}$.

☆

$\Rightarrow$ All paths (above or below) have the same phase, $\int d\vec{x} \cdot \vec{A}$,

but. the paths above have a different phase from the ones below.

$$\Rightarrow K(B,A) = K_{\uparrow}^{(0)} \exp\left(\frac{ie}{\hbar c}\int_{l_\uparrow} d\vec{x}\cdot\vec{A}\right) + K_{\downarrow}^{(0)} \exp\left(\frac{ie}{\hbar c}\int_{l_\downarrow} d\vec{x}\cdot\vec{A}\right)$$

($\uparrow$: above, $\downarrow$: below)

$\therefore$ transition amplitude $\langle B|A\rangle = K(B,A)$

$$\propto \left[1 + C_0 \exp\left(\frac{ie}{\hbar c}\Phi_B\right)\right]$$

: Oscillation with a period $\frac{2\pi\hbar c}{|e|}$ by increasing Φ_B.

$$\frac{2\pi\hbar c}{|e|} = 4.135 \times 10^{-7} \text{ gauss-cm}^2$$

: a fundamental unit of magnetic flux.

- Example 2.: Magnetic Monopole.

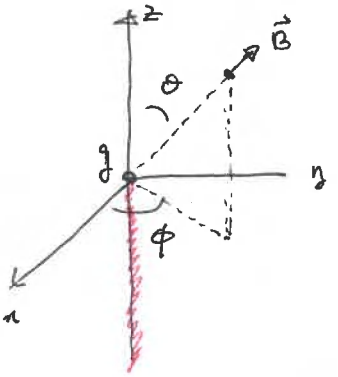
If there exists a magnetic monopole,

$$\nabla\cdot\vec{B} = 4\pi\rho_m \longleftrightarrow \nabla\cdot\vec{E} = 4\pi\rho.$$

A point magnetic monopole at the origin generates

$$\vec{B} = \frac{g}{r^2}\hat{r} \qquad \| \; g : \text{a point magnetic charge.}$$

which corresponds to $\vec{A} = g\,\dfrac{1-\cos\theta}{r\sin\theta}\,\hat{\phi}$



$\hookrightarrow$ $\boxed{\text{It's singular at } \theta = \pi.}$!

Q. Can this be inevitable? ... Yes.

Q. Is there any way to detour it?

1. Dirac string
2. Two vector potentials.

- The singularity is essential.

The Gauß's law : $\int_S \vec{B} \cdot d\vec{\sigma} = 4\pi g$, $\vec{B} = \nabla \times \vec{A}$

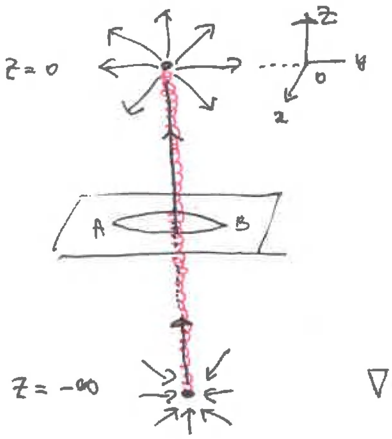
The Divergence theorem : $\int_S \vec{B} \cdot d\vec{\sigma} = \int_V da\, \nabla \cdot (\nabla \times \vec{A})$

$$\begin{array}{ccc} \| & & \| \\ 4\pi g & \neq & 0 \end{array}$$ (if A is non-singular)

→ "$\vec{A}$ is singular, if there's magnetic monopole."

- How can we "detour" the singularity

① Dirac string : put an infinitesimally thin and semi-infinitely long solenoid to replace the singularity at $z < 0$.



$$\vec{B}_{\text{string}}(x,y,z) = 4\pi g\, \delta(x)\, \delta(y)\, [1 - \Theta(z)]\, \hat{z}$$

$$\nabla \cdot (\underset{\uparrow\ \text{monopole}}{\vec{B}_g} + \vec{B}_{\text{string}}) = 0 \,,$$ no singularity.

But, this string is virtual, undetectable!

Test of the AB effet : $\Phi_B = \int \vec{B}_{\text{string}} \cdot d\vec{\sigma}$
$= 4\pi g$.

⇒ phase diff. $= \frac{|e|}{\hbar c} 4\pi g$

⇒ $2\pi n$ (if it's undetectable.)

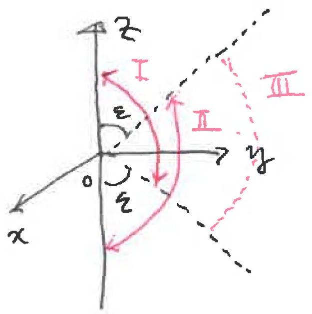
∴ The smallest magnetic charge

$$g = \frac{\hbar c}{2|e|}$$ and, it's quantized.

→ If there's a magnetic monopole,

$$|e| = \frac{\hbar c}{2g} \cdot n$$: electronic charge is quantized!

② Two vector potentials



$$\mathrm{I}: \vec{A}^{(\mathrm{I})} = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\phi} \quad (\theta < \pi - \varepsilon)$$

↳ singular at $\theta = \pi$.

$$\mathrm{II}: \vec{A}^{(\mathrm{II})} = -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{\phi} \quad (\theta > \varepsilon)$$

↳ singular at $\theta = 0$.

- Both of $A^{(\mathrm{I})}$ and $A^{(\mathrm{II})}$ produces $\vec{B} = \frac{g}{r^2}\hat{r}$ for their valid θ.

- In area III ($\varepsilon < \theta < \pi - \varepsilon$), both are valid.

→ Trouble: we expect a single-valued wave function. (No branch cut!)

↳ Since $\vec{A}^{(\mathrm{II})} - \vec{A}^{(\mathrm{I})} = -\frac{2g}{r\sin\theta}\hat{\phi}$,

the gauge transformation is written as

$$\vec{A}^{(\mathrm{II})} = \vec{A}^{(\mathrm{I})} + \nabla\Lambda, \quad \Lambda = -2g\phi.$$

Thus,
$$\psi^{(\mathrm{II})} = \exp\left(\frac{-2ieg\phi}{\hbar c}\right)\psi^{(\mathrm{I})}.$$

They are well-defined for $\phi = (0, 2\pi)$ or $\phi \to \phi + 2\pi$,

when
$$\frac{2|e|g\cdot 2\pi}{\hbar c} = 2\pi n.$$

$$\Rightarrow g = \frac{\hbar c}{2|e|}\cdot n \simeq \left(\frac{137}{2}\right)|e|\cdot n$$

$\|$ (fine structure const)$^{-1}$ $\frac{\hbar c}{|e|^2} \simeq 137$

or
$$|e| = \frac{\hbar c}{2g}\cdot n.$$

The same conclusions without the Dirac strings.

QM does not reject the magnetic monopole.

⇒ There're artifical quantum systems (spin ICE, spin-1 BEC): analogs of the monopole.